FREE TRIAL

SEQUOIA
— EDUCATION —

Write your **full name** in the space above.

Specialist Mathematics Examination 2

Question and Answer Book

VCE Trial Examination – Free

- Reading time is **15 minutes**
- Writing time is **2 hours**

Approved materials

- Protractors, set squares and aids for curve sketching
- One bound reference
- One approved CAS calculator or CAS software, and one scientific calculator

Materials supplied

- Question and Answer Book of 24 pages
- Formula Sheet
- Multiple-Choice Answer Sheet

Instructions

- Follow the instructions on your Multiple-Choice Answer Sheet.
- At the end of the examination, place your Multiple-Choice Answer Sheet inside the front cover of this book.

Students are **not** permitted to bring mobile phones and/or any unauthorised electronic devices into the examination room.

Contents pages

Section A – Multiple-choice questions

Instructions

- Answer **all** questions in pencil on your Multiple-Choice Answer Sheet.
- Choose the response that is **correct** for the question.
- A correct answer scores 1; an incorrect answer scores 0.
- Marks will **not** be deducted for incorrect answers.
- No marks will be given if more than one answer is completed for any question.
- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
- Take the **acceleration due to gravity** to have magnitude g m s^{-2}, where $g = 9.8$

Question 1

Consider the following statement.

'If an animal is a herbivore, then it does not eat meat.'

Which one of the following options is the converse of this statement?

A. If an animal is not a herbivore, then it eats meat.

B. If an animal eats meat, then it is not a herbivore.

C. If an animal is not a herbivore, then it does not eat meat.

D. If an animal does not eat meat, then it is a herbivore.

Question 2

Suppose that $\sin(x) = \dfrac{12}{13}$ and $\cos(y) = \dfrac{4}{5}$, where $x, y \in \left(0, \dfrac{\pi}{2}\right)$.

Then $\sin(x - y)$ is equal to

A. $-\dfrac{63}{65}$

B. $-\dfrac{33}{65}$

C. $\dfrac{33}{65}$

D. $\dfrac{63}{65}$

Question 3

Let $f : \mathbb{R} \to \mathbb{R},\ f(x) = |x-a| - |x-b|$, where a and b are real constants such that $a \leq b$.

The range of f is

A. $[a-b,\ -a+b]$

B. $[-a-b,\ a+b]$

C. $[-a+b,\ \infty)$

D. $[a-b,\ \infty)$

Question 4

Consider the following pseudocode. Here, `i` is the imaginary unit and `abs(z)` calculates the complex modulus of `z`.

```
define inMandelbrot(c, max)
    # c:  a complex number
    # max:  a positive integer
    z ← 0
    for n from 1 to max do:
        z ← z^2 + c
        if abs(z) > 2 then
            return False
        end if
    end for
    return True
```

The smallest positive integer `max` for which

```
inMandelbrot(1/2 + i/2, max)
```

returns `False` is

A. 3

B. 4

C. 5

D. 6

Question 5

Let $w = \text{cis}\left(\frac{\pi}{3}\right)$.

The sum $w^{1!} + w^{2!} + w^{3!} + \cdots + w^{2025!}$ is equal to

A. 2023

B. 2025

C. $2023 + \sqrt{3}i$

D. $2025 + \sqrt{3}i$

Question 6

Let $p(z)$ be a polynomial over $\mathbb{C}$ with real coefficients such that $(z - 2 + i)^2$ and $z + 3i$ are factors.

The minimum degree of $p(z)$ is

A. 3

B. 4

C. 5

D. 6

Question 7

A spherical balloon with radius r cm is being inflated at a rate of 2 cm^3 s^{-1}.

The rate, in cm s^{-1}, at which the radius of the balloon is increasing is given by

A. $\dfrac{1}{8\pi r^2}$

B. $\dfrac{1}{2\pi r^2}$

C. $2\pi r^2$

D. $8\pi r^2$

Question 8

Consider the sequence $I_n = \int_0^1 x^n e^{-x}\,dx$, where n is a non-negative integer.

A recursion relation for this sequence, for $n \geq 1$, is given by

A. $I_n = e^{-1} - nI_{n-1}$

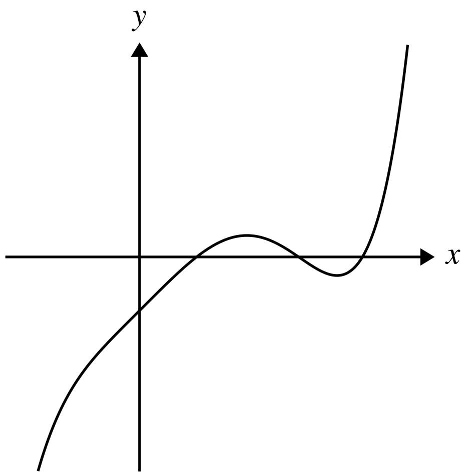
B. $I_n = -e^{-1} + nI_{n-1}$

C. $I_n = -nI_{n-1}$

D. $I_n = nI_{n-1}$

Question 9

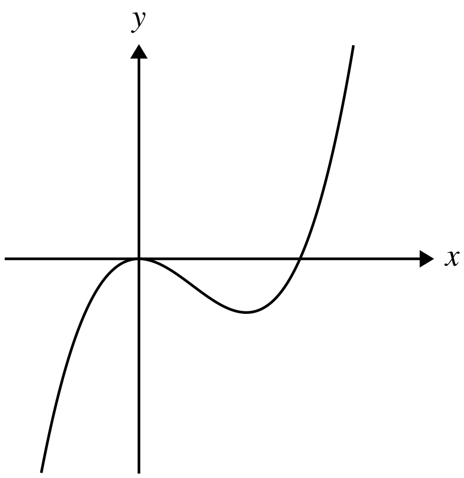
The graph of a twice-differentiable function $f : \mathbb{R} \to \mathbb{R}$ is shown below.



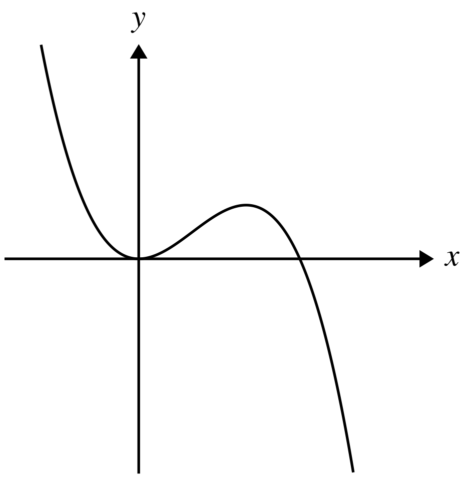
All axes below have the same scale as those in the diagram above.

Which one of the following options best represents the graph of the **second** derivative of f?

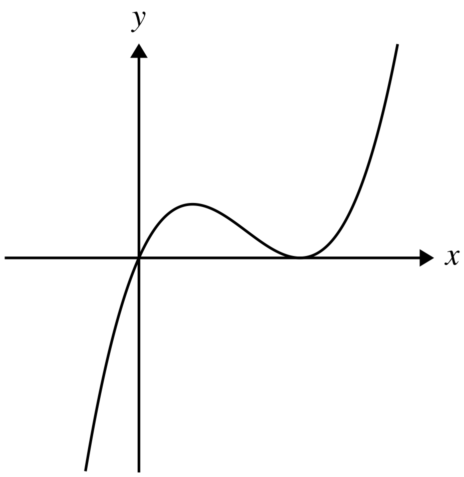
A.



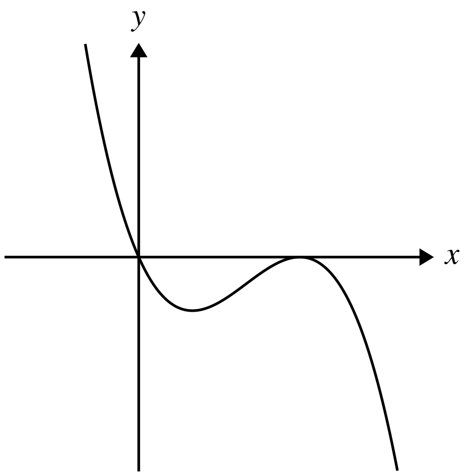
B.



C.



D.



Question 10

Consider the differential equation $\frac{dy}{dx} = x^2 - \sqrt{y}$, where $y(0) = 1$.

Using Euler's method with a step size of 0.1, an approximation of $y(0.2)$ is

A. 0.8010

B. 0.8045

C. 0.8061

D. 0.8101

Question 11

A tank initially contains 10 kg of salt dissolved in 50 L of water. A 0.1 kg L^{-1} saltwater solution flows into the tank at 10 L min^{-1}. The mixture in the tank, which is kept uniform by stirring, flows out at 5 L min^{-1}.

If x kilograms is the amount of salt in the tank after t minutes, the differential equation relating x and t is

A. $\frac{dx}{dt} + \frac{x}{10+t} = 1$

B. $\frac{dx}{dt} + \frac{x}{10+t} = \frac{1}{10}$

C. $\frac{dx}{dt} + \frac{x}{50+5t} = 1$

D. $\frac{dx}{dt} + \frac{x}{50+5t} = \frac{1}{10}$

Question 12

A particle is travelling in a straight line such that its velocity, v m s^{-1}, is given by $v^2 = 36 - x^2$, where x is its displacement, in metres, from an origin O.

The acceleration, in m s^{-2}, of the object where $x = -2$ is

A. -4

B. -2

C. 2

D. 4

Question 13

Consider the vectors $\underset{\sim}{a} = -\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}$, $\underset{\sim}{b} = 2\underset{\sim}{i} - \underset{\sim}{j} - \underset{\sim}{k}$ and $\underset{\sim}{c} = m\underset{\sim}{i} + n\underset{\sim}{j}$, where m and n are real constants.

The vectors are linearly dependent if and only if

A. $m = n$

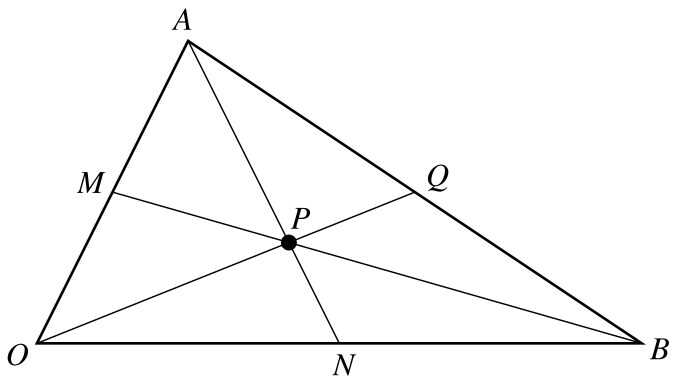
B. $m \neq n$

C. $m = -n$

D. $m \neq -n$

Question 14

Let OAB be a triangle with $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OB} = \underset{\sim}{b}$. Let M, N and Q be the midpoints of OA, OB and AB respectively. The medians OQ, AN and BM intersect at a single point P, as shown below.



The vector $\overrightarrow{OP}$ is equal to

A. $\frac{1}{3}\underset{\sim}{a} + \frac{1}{3}\underset{\sim}{b}$

B. $\frac{1}{3}\underset{\sim}{a} + \frac{2}{3}\underset{\sim}{b}$

C. $\frac{2}{3}\underset{\sim}{a} + \frac{1}{3}\underset{\sim}{b}$

D. $\frac{2}{3}\underset{\sim}{a} + \frac{2}{3}\underset{\sim}{b}$

Question 15

The distance between the planes given by $x-4y+8z=4$ and $-x+4y-8z=2$ is

A. $\frac{2}{9}$

B. $\frac{4}{9}$

C. $\frac{2}{3}$

D. $\frac{8}{9}$

Question 16

The displacement, in metres, from an origin O, of a drone t seconds after leaving the ground is given by

$$\underset{\sim}{r}(t)=\sin(t)\,\underset{\sim}{i}+\cos(t)\,\underset{\sim}{j}+t\,\underset{\sim}{k},\quad t\geq 0,$$

where $\underset{\sim}{i}$ and $\underset{\sim}{j}$ are perpendicular horizontal unit vectors and $\underset{\sim}{k}$ is the unit vector pointing upwards.

After two seconds, the angle of elevation of the drone from O is closest to

A. $27°$

B. $42°$

C. $48°$

D. $63°$

Question 17

Consider the lines L_1 and L_2 given by

$$L_1:\ \underset{\sim}{r}_1(s)=-\underset{\sim}{i}+4\underset{\sim}{j}+s\left(2\underset{\sim}{j}-\underset{\sim}{k}\right)\quad\text{and}\quad L_2:\ \underset{\sim}{r}_2(t)=2\underset{\sim}{i}+\underset{\sim}{k}+t\left(3\underset{\sim}{i}-\underset{\sim}{k}\right),$$

where $s,t\in\mathbb{R}$.

The lines L_1 and L_2 intersect at

A. $(2,0,1)$

B. $(-4,0,3)$

C. $(1,-2,1)$

D. $(-1,0,2)$

Question 18

A tetrahedron is formed with vertices O, A, B and C, where $\overrightarrow{OA} = \underset{\sim}{a}$, $\overrightarrow{OB} = \underset{\sim}{b}$ and $\overrightarrow{OC} = \underset{\sim}{c}$.

The volume of the tetrahedron is given by

A. $\frac{1}{6}\,|\underset{\sim}{a}|\;|\underset{\sim}{b} \times \underset{\sim}{c}|$

B. $\frac{1}{6}\,|\underset{\sim}{a} \cdot (\underset{\sim}{b} \times \underset{\sim}{c})|$

C. $\frac{1}{3}\,|\underset{\sim}{a}|\;|\underset{\sim}{b} \times \underset{\sim}{c}|$

D. $\frac{1}{3}\,|\underset{\sim}{a} \cdot (\underset{\sim}{b} \times \underset{\sim}{c})|$

Question 19

Let $X \sim N\left(15, 5^2\right)$ and $Y \sim N\left(10, 10^2\right)$ be independent random variables.

If Z is the standard normal random variable, then $\Pr(3X > 2Y)$ is equal to

A. $\Pr\left(Z > -\frac{5}{3}\right)$

B. $\Pr(Z > -1)$

C. $\Pr\left(Z > -\frac{5}{7}\right)$

D. $\Pr(Z > 0)$

Question 20

In a statistical test, a type II error occurs if

A. H_0 is rejected when H_0 is true.

B. H_0 is rejected when H_0 is false.

C. H_0 is not rejected when H_0 is true.

D. H_0 is not rejected when H_0 is false.

End of Section A

Section B

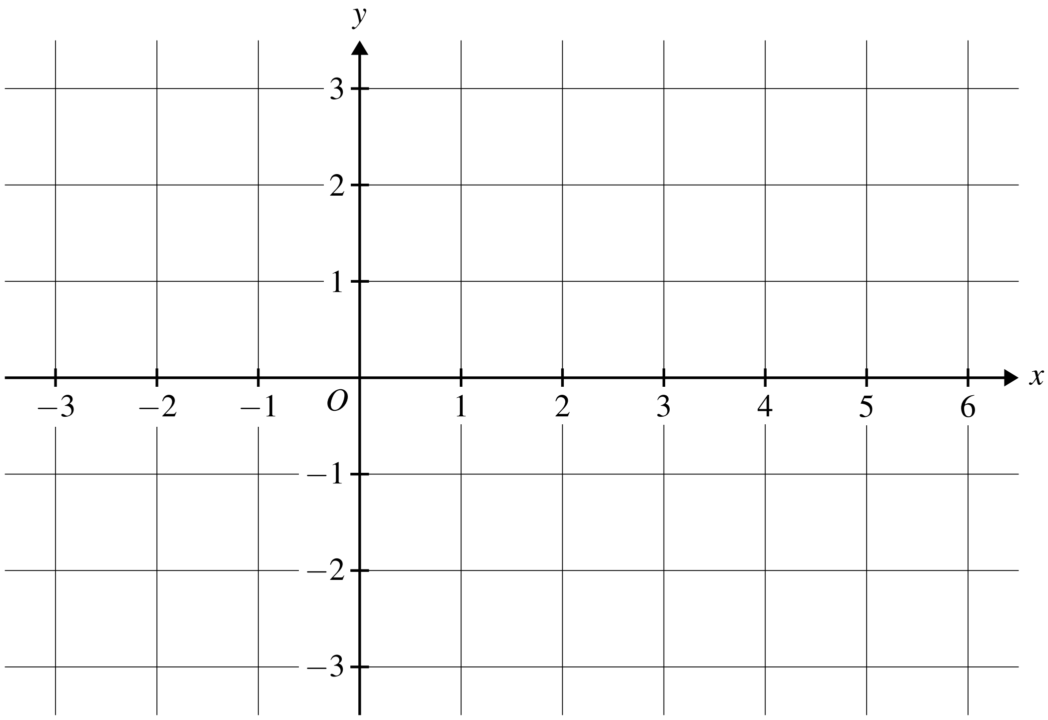
Instructions

- Answer **all** questions in the spaces provided.
- Write your responses in English.
- Unless otherwise specified, an **exact** answer is required to a question.
- In questions where more than one mark is available, appropriate working **must** be shown.
- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
- Take the **acceleration due to gravity** to have magnitude $g \text{ m s}^{-2}$, where $g = 9.8$

Question 1 (11 marks)

Consider the function $f(x) = \dfrac{2x}{x^2 - 2x + 2}$.

a. Sketch the graph of f on the axes below. Label any asymptotes with their equations and label the stationary points with their coordinates. 3 marks



b. The region enclosed by the graph of f, the x-axis, and the line $x = 2$ is rotated about the x-axis to form a solid of revolution.

Find the volume of this solid. 2 marks

c. Find the coordinates of the points of inflection of the graph of f. 2 marks

Consider the function $g(x) = \dfrac{2x}{x^2 + kx + 2}$, where k is a real constant.

d. Find the values of k for which the graph of g has exactly one vertical asymptote. 1 mark

e. The second derivative of g is given by $g''(x) = \dfrac{4\left(x^3 - 6x - 2k\right)}{\left(x^2 + kx + 2\right)^3}$.

Find the values of k for which the graph of g has exactly three points of inflection. 3 marks

Do not write in this area.

Question 2 (10 marks)

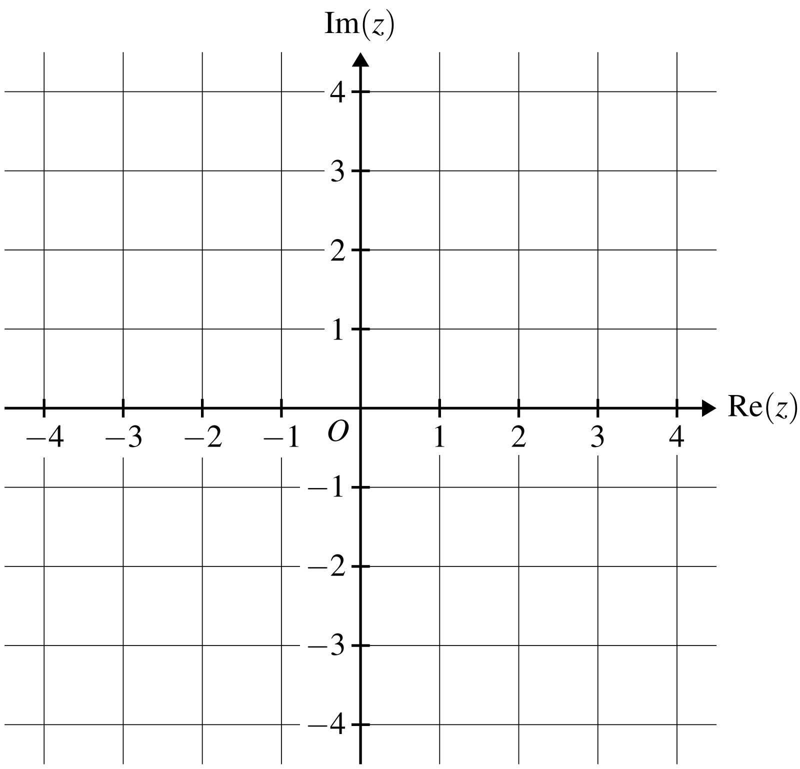
Consider the polynomial $p(z) = z^2 - 6z + 12$, where $z \in \mathbb{C}$.

a. Show that the roots of $p(z)$ are $3 \pm \sqrt{3}i$. 1 mark

Consider the circle given by $S = \{z \in \mathbb{C} : |z - 2| = 2\}$.

b. Show that the roots of $p(z)$ lie on the circle S. 1 mark

c. Sketch S and label the roots of $p(z)$ on the Argand diagram below. 2 marks



The line $L = \{z \in \mathbb{C} : |z| = |z-6|\}$ divides the circle S into two segments.

d. **i.** Sketch L on the Argand diagram in **part c**. 1 mark

ii. Find the area of the minor segment of the circle S bounded by the line L. 2 marks

Consider the polynomial $q(z) = az^2 + bz + c$, where $a, b, c \in \mathbb{R} \setminus \{0\}$ and $b^2 - 4ac < 0$.

e. Solve $q(i\bar{z}) = 0$ for $z \in \mathbb{C}$. Express your solutions in Cartesian form, in terms of a, b and c. 1 mark

f. Suppose the line defined by $\{z \in \mathbb{C} : |z| = |z-w|\}$, where $w \in \mathbb{C}$, passes through the solutions of the equation $q(i\bar{z}) = 0$.

Find w in terms of a, b and c. 2 marks

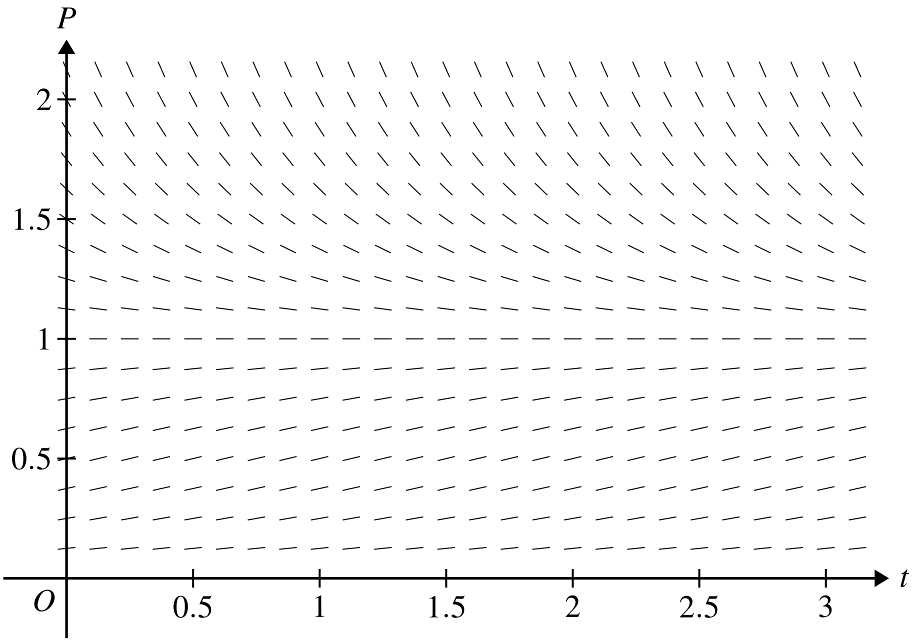
Question 3 (10 marks)

Consider the logistic differential equation

$$\frac{dP}{dt} = P(1-P), \quad P(0) = P_0 > 0,$$

where P represents a population at time t.

a. On the slope field below, sketch the solution curves of the differential equation corresponding to the initial conditions $P_0 = 0.25$, $P_0 = 0.75$ and $P_0 = 1.5$. 3 marks



b. State the value of P_0 for which the differential equation has a constant solution. 1 mark

c. **i.** Use implicit differentiation to show that $\frac{d^2P}{dt^2} = P(1-P)(1-2P)$. 1 mark

ii. Find the set of values P_0 for which the solution curve has a point of inflection. 1 mark

d. **i.** Solve the logistic differential equation with the initial condition $P_0 = 0.25$, expressing P as a function of t. 3 marks

ii. Find the t-coordinate of the point of inflection for this particular solution curve. 1 mark

Do not write in this area.

Question 4 (10 marks)

The displacements of two particles, A and B, after t seconds are given by

$$\underset{\sim}{r}_A(t) = \left(3t - t^2\right)\underset{\sim}{i} + (1-t)\underset{\sim}{j} \quad \text{and} \quad \underset{\sim}{r}_B(t) = (2t-1)\underset{\sim}{i} + t\underset{\sim}{j}, \quad t \geq 0,$$

where displacement components are measured in metres.

a. Show that the particles do not collide. 2 marks

b. Find the Cartesian equation of the path for each particle. 2 marks

c. Find the coordinates of the point where the paths of the particles cross, correct to two decimal places. 1 mark

Do not write in this area.

d. Find the minimum distance, in metres, between the two particles, and the time when it occurs. 3 marks

e. Find the times at which the particles have equal speeds. 2 marks

Do not write in this area.

Question 5 (10 marks)

A plane P in $\mathbb{R}^3$ is parameterised by

$$x = s - t$$
$$y = 2 - 3s - 3t$$
$$z = 2 + 2s - 2t$$

where $s, t \in \mathbb{R}$.

a. Show that the plane P has Cartesian equation $-2x + z = 2$, where $x, z \in \mathbb{R}$. 2 marks

b. Which coordinate plane is P perpendicular to? Justify your answer. 1 mark

Consider the line L in $\mathbb{R}^3$ given by vector equation

$$\underset{\sim}{r}(t) = \underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k} + t(-2\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}), \quad t \in \mathbb{R}.$$

c. Find the coordinates of the point where the line L intersects the plane P. 2 marks

Do not write in this area.

d. Express $-2\underset{\sim}{i}+\underset{\sim}{j}+\underset{\sim}{k}$ as the sum of two orthogonal vectors: one parallel to the plane P and the other perpendicular to P. 3 marks

e. Hence or otherwise, find a vector equation of L', the reflection of the line L in the plane P. 2 marks

Do not write in this area.

Question 6 (9 marks)

The heights of fully grown sweet corn plants on a farm are known to be normally distributed with a mean of 3.2 metres and a standard deviation of 0.45 metres.

From a large plantation, two fully grown sweet corn plants are randomly selected. Assume that the heights of the plants are independent.

a. Find the probability that the mean height of the two plants exceeds 3.1 metres. Give your answer correct to four decimal places. 1 mark

b. By visual inspection, the heights of the two plants differ by more than 10 centimetres.

Given this, find the probability that their heights differ by less than 20 centimetres. Give your answer correct to four decimal places. 3 marks

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To reduce production costs, the farm is trialling a new, cheaper fertiliser.

To test whether the new fertiliser affects the mean height of sweet corn plants once fully grown, a sample of 60 plants is grown using the new fertiliser. The mean height of these plants is found to be 3.3 metres.

A two-tailed hypothesis test is to be conducted at the 5% level of significance. Assume that the heights of fully grown sweet corn plants remain normally distributed with a standard deviation of 0.45 metres.

c. State suitable null and alternative hypotheses for this test. 1 mark

d. Find the p value for the test, correct to four decimal places. 1 mark

e. Does the sample provide evidence at the 5% level of significance that the new fertiliser changes the mean height of the plants? Justify your answer. 1 mark

f. Find the critical sample means for the test. Give your answer in metres, correct to four decimal places. 2 marks

End of Question and Answer Book

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